Math 254B Lecture 21 Notes

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1 Frostman's Lemma, Product Dimension, Slices, and Projections

1.1 Frostman's lemma

Let (X, ρ) be a compact metric space. We have shown the following.

Proposition 1.1. *If* $\sum_{i \geq 1} c_i \mathbb{1}_{E_i} \geq \mathbb{1}_X$ *and* $\operatorname{diam}(E_i) < \delta$ *for all* i*, then*

$$\sum_i c_i (\operatorname{diam}(E_i))^\alpha \geq O_\alpha(1) \cdot \mathcal{H}_\delta^\alpha(X).$$

We used this to prove Frostman's lemma:

Lemma 1.1 (Frostman)**.** *If* $m_\alpha(X) > 0$ *, then there s a measure* $\mu \in P(X)^\alpha$ *and* $c < \infty$ *such that* $\mu(B_r(x)) \leq cr^\alpha$ *for all* x, r*.*

Let's go over the proof again, more carefully.

Proof. Pick $\delta > 0$ such that $\mathcal{H}_\alpha^\delta(X) > 0$. On $C(X)$, define

$$p(f) := \inf \left\{ \sum_{i \geq 1} c_i (\operatorname{diam}(E_i))^\alpha : \sum_{i \geq 1} c_i \mathbb{1}_{E_i} \geq f, \operatorname{diam}(E_i) < \delta, c_i \geq 0 \right\}.$$

Observe:

- We can pick $E_i = X$, so $p(f) < \infty$.
- If $f \leq 0$, then $p(f) = 0$.
- $p(tf) = tp(f)$ if $t \geq 0$.
- $p(f + g) \leq p(f) + p(g)$

- $p(\mathbb{1}_X) > 0$ (bounded below by $wm_\alpha(X)$).

Define $\ell \in (\mathbb{R} \cdot \mathbb{1}_X)^*$ by $\ell(\mathbb{1}_X) = p(\mathbb{1}_X)$. By Hahn-Banach and Riesz-representation, there is a a bounded linear functional (i.e. a measure) $\mu \in C(X)^*$ such that $|\int f \, d\mu| \leq p(f)$ for all f and $\mu(X) = p(\mathbb{1}_X)$. Note that if $f \leq 0$, then $\int f \, d\mu \leq p(f) = 0$. So $\mu \geq 0$. After normalizing by $p(\mathbb{1}_X)$, we may assume that $\mu \in P(X)$. Let $B_r(x)$. Then $\mu(B_r(x)) = \sup\{\int f \, d\mu : f \in C(X), f \geq 0, f \leq 1_{B_r(x)}\}$. For any such f, $p(f) \leq (2r)^\alpha$ (up to a normalization constant). $\square$

Remark 1.1. In this proof, the Hahn-Banach theorem is being used as a sort of infinite-dimensional linear optimization.

1.2 Dimension of products

Theorem 1.1. *Let (X, ρ_X) and (Y, ρ_Y) be compact metric spaces. Let $A \in \mathcal{B}_X$ and $B \in \mathcal{B}_Y$. Then*

$$\dim(A) + \dim(B) \leq \dim(A \times B) \leq \dim(A) + \overline{\dim}_B(B).$$

Corollary 1.1. *If $\dim(A) = \overline{\dim}_B(A)$, then these are all equalities.*

Remark 1.2. This inequalities can be strict.

Proof. Proof of the left inequality: We will actually show that if $m_\alpha(A) > 0$ and $m_\beta(B) > 0$, then $m_{\alpha+\beta}(A \times B) > 0$. Assume that A anf B are compact.[1] Then there exist measures $\mu \in P(A)$and $\nu \in P(B)$ such that $\mu(B_r^X(x)) \leq cr^\alpha$ for all $r \leq r_0^x$ and $\nu(B_r^Y(y)) \leq cr^\beta$ for ll $r \leq r_0^y$. Let $\lambda = \mu \times \nu$. Then equip $X \times Y$ with the box metric

$$\rho((x,y),(x',y')) := \max\{\rho_X(x,x'), \rho_Y(y,y')\}.$$

We get

$$\lambda(B_r^{X\times Y}(x,y) = \lambda(B_r^X(x) \times B_r^Y(y)) \leq c^2 r^| alpha r^\alpha, \qquad \forall r \leq \min\{r_0^X, r_0^Y\}.$$

By the mass distribution principle, $\dim(A \times B) \geq \alpha + \beta$.

Proof of the right inequality: Let $\alpha > \dim(A)$ and $\beta > \overline{\dim}_B(B)$. Then there exists $(E_i)_i$ such that $A \subseteq \bigcup_i E_i$ and $\sum_i (\mathrm{diam}(E_i))^| alpha < \varepsilon$. The idea is to cover $A \times B$ by strips with one side E_i and to find nice sets within these strips: For each i, choose a covering $\mathcal{F}_i$ of B by sets of diameter $\leq \mathrm{diam}(E_i)$ such that $|\mathcal{F}_i| \leq \mathrm{diam}(E_i)^{-\beta}$. Consider $\mathcal{E} = \{E_i \times F : i \geq 1, F \in \mathcal{F}_i\}$; note that $\mathrm{diam}(E_i \times F) \leq \mathrm{diam}(E_i)$ by the definition of the box metric. Now $A \times B \subseteq \bigcup \mathcal{E}$, and

$$\sum_{(E_i,F)\in\mathcal{E}} (\mathrm{diam}(E_i))^{\alpha+\beta} = \sum_i (\mathrm{diam}(E_i))^\alpha |\mathcal{F}_i| (\mathrm{diam}(E_i))^\beta \leq \sum_i (\mathrm{diam}(E_i))^\alpha. \qquad \square$$

[1] This assumption is only here because we have only proven Frostman's lemma in this special case. Frostman's lemma is true for more general spaces.

1.3 Slices and projections of α-regular sets

Definition 1.1. Let μ be a positive Borel measure on (X, ρ). Call it α**-regular** if there exists some $r_0 > 0$ and $c < \infty$ such that $\mu(B_r(x)) \leq cr^\alpha$ for all $r \leq r_0$.

Theorem 1.2. *Let (X, ρ_X) and (Y, ρ_Y) be compact metric spaces, and let $A \subseteq X \times Y$. Let $0 \leq \alpha \leq \beta$, and assume μ is α-regular on Y. For all $y \in Y$, let $L_y = X \times \{y\}$. Then*

$$\int m_{\beta-\alpha}(A \cap L_y)\, d\mu(y) \leq \text{const} \cdot m_\beta(A).$$

Here is an important special case:

Corollary 1.2. *If $X = \mathbb{R}^n$. $Y = \mathbb{R}^m$, μ is Lebesgue measure, and $\alpha = m$, then if $A \subseteq \mathcal{B}_{R^{n\times m}}$, then* $\dim(A \cap L_y) \leq \max\{0, \dim(A) - m\}$ *for a.e. y.*

Proof. Let $A \subseteq \bigcup_i E_i \times F_i$ with $\sum_i \text{diam}(E_i \times F_i)^\beta \leq C$. For any y, we have

$$A \cap L_y \subseteq \bigcup_{\{i : y \in F_i\}} E_i.$$

Now

$$\begin{aligned} m_{\beta-\alpha}(A \cap L_y)\, d\mu(y) &\leq \sum_i (\text{diam}(E_i))^{\beta-\alpha} \underbrace{\int \mathbb{1}_{F_i}(y)\, d\mu(y)}_{=\mu(F_i) \leq \text{const} \cdot \text{diam}(F_i)^\alpha} \\ &\leq \text{const} \sum_i (\text{diam}(E_i \times F_i))^{\beta-\alpha+\alpha} \\ &\leq \text{const} \cdot m_\beta(A). \end{aligned}$$

□

Example 1.1. Let $C_\alpha \subseteq \mathbb{R}$ with $d = \dim(C_\alpha) = d(\alpha)$. Then the Cantor dust set $C_\alpha \times C_\alpha \subseteq \mathbb{R}^2$. The dimensions agree for this set, so $\text{diam}(C_\alpha \times C_\alpha) = 2d$. IF $y \in C_\alpha$, then $(C_\alpha \times C_\alpha) \cap L_y = C_\alpha$, so $\dim = d > 2d - 1$. So maybe we should not be looking at this with Lebesgue measure; we should look at it using a measure defined on the Cantor set.

Later, we will meet fractals coming from dynamics, where the exceptional set is empty except in a few directions.

There is also a dual notion to measuring slices: measuring projections.

Theorem 1.3. *If $F \in \mathcal{B}_{\mathbb{R}^2}$ and P_θ is a line in direction θ, then* $\dim(P_\theta F) = \min\{1, \dim(F)\}$ *for a.e. θ.*